

CP violation in neutrino mixing with $\delta = -\pi/2$ in A_4 Type-II seesaw model

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Abstract

We study a class of models for neutrino mass matrix in Type-II seesaw with A_4 family symmetry. The resulting neutrino mass matrix can be naturally made to respect a $\mu - \tau$ exchange plus CP conjugate symmetry (GLS) with the CP violating phase δ and the mixing angle θ_{23} predicted to be $\pm\pi/2$ and $\pi/4$, respectively. When GLS is explicitly broken by complex Yukawa couplings, the model predictions for δ and θ_{23} can be significantly modified. Should future experiments will indeed determine θ_{23} and δ_{CP} away from the GLS limit values, one then had to consider models with broken GLS. We study several simple scenarios to show how the modifications arise when GLS is broken and how future experiments can test this class of models.

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I. INTRODUCTION

Experiments have collected a lot of precious information about the neutrino mixing parameters. The mixing angles in the Pontecorvo-Maki-Nakagawa-Sakata matrix[1] V_{PMNS} are not always small [2–4] as their quark mixing counter part[2, 5]. In the standard parametrization[2, 6] for three neutrino mixing commonly used[3, 4], the mixing angle θ_{23} is close to $\pi/4$, θ_{12} is also large, θ_{13} is relatively small but away from zero. There are also evidences showing that the CP violating Dirac phase δ is close to $-\pi/2$ (or $3\pi/2$). If these data are further confirmed, the neutrino mass matrix will have a simple form. Assuming that neutrinos are Majorana particles, the neutrino mass matrix defined by the term giving neutrino masses in the Lagrangian $(1/2)\bar{\nu}_L m_\nu \nu_L^c$ has the following form,

$$m_\nu = V_{PMNS} \hat{m}_\nu V_{PMNS}^T, \quad (1)$$

where $\hat{m}_\nu = \text{diag}(m_1, m_2, m_3)$ with $m_i = |m_i| \exp(i\alpha_i)$. Here we have put Majorana phase information in the neutrino masses. The standard form for V_{PMNS} is given by

$$V_{PMNS} = U(\theta_{12}, \theta_{13}, \theta_{23}, \delta) = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13} \end{pmatrix}, \quad (2)$$

where c_{ij} and s_{ij} are $\cos \theta_{ij}$ and $\sin \theta_{ij}$, respectively. They are all normalized to be positive.

With $\delta = -\pi/2$ and $\theta_{23} = \pi/4$, m_ν has the following form[7, 9, 10]

$$m_\nu = \begin{pmatrix} a & c + i\beta & -(c - i\beta) \\ c + i\beta & d + i\gamma & \tilde{b} \\ -(c - i\beta) & \tilde{b} & d - i\gamma \end{pmatrix}, \quad (3)$$

where

$$\begin{aligned} a &= m_1 c_{12}^2 c_{13}^2 + m_2 s_{12}^2 c_{13}^2 - m_3 s_{13}^2, & \tilde{b} &= -\frac{1}{2} (m_1 (s_{12}^2 + c_{12}^2 s_{13}^2) + m_2 (c_{12}^2 + s_{12}^2 s_{13}^2) - m_3 c_{13}^2), \\ c &= -\frac{1}{\sqrt{2}} (m_1 - m_2) s_{12} c_{12} c_{13}, & d &= \frac{1}{2} (m_1 (s_{12}^2 - c_{12}^2 s_{13}^2) + m_2 (c_{12}^2 - s_{12}^2 s_{13}^2) + m_3 c_{13}^2), \\ \beta &= \frac{1}{\sqrt{2}} s_{13} c_{13} (m_1 c_{12}^2 + m_2 s_{12}^2 + m_3), & \gamma &= -(m_1 - m_2) s_{12} c_{12} s_{13}. \end{aligned} \quad (4)$$

Note that in the most general case, because non-zero Majorana phases, the parameters a , \tilde{b} , c , d , β and γ are all complex.

One has the degrees of freedom to redefine the neutrino fields phases and the most general form of the above mass matrix can be rewritten as

$$m_\nu = \begin{pmatrix} e^{ip_1} & 0 & 0 \\ 0 & e^{ip_2} & 0 \\ 0 & 0 & e^{ip_3} \end{pmatrix} \begin{pmatrix} a & c + i\beta & -(c - i\beta) \\ c + i\beta & d + i\gamma & \tilde{b} \\ -(c - i\beta) & \tilde{b} & d - i\gamma \end{pmatrix} \begin{pmatrix} e^{ip_1} & 0 & 0 \\ 0 & e^{ip_2} & 0 \\ 0 & 0 & e^{ip_3} \end{pmatrix}, \quad (5)$$

where the phases p_i are arbitrary. One can choose some particular values for p_i to obtain forms of m_ν for convenience of analysis. For example the “-” sign for the “13” and “31” entries can be

removed by choosing $p_1 = p_2 = 0$ and $p_3 = \pi$, the resultant matrix can be written in a more familiar forms

$$m_\nu = \begin{pmatrix} a & c + i\beta & (c - i\beta) \\ c + i\beta & d + i\gamma & b \\ (c - i\beta) & b & d - i\gamma \end{pmatrix}, \quad (6)$$

where $b = -\tilde{b}$.

If neutrinos do not have non-trivial Majorana phases (but mass can be positive or negative), the mass matrix has the following form

$$m_\nu = \begin{pmatrix} A & C & C^* \\ C & D^* & B \\ C^* & B & D \end{pmatrix}. \quad (7)$$

Replacing $\delta = -\pi/2$ by $\delta = \pi/2$, the neutrino mass matrix is given in a similar form as that in eq.(6), but β and γ need to be multiplied by a “-” sign. This implies that without further information given, a general mass matrix in the form given by eq.(6) can give $\delta = \pm\pi/2$ and $\theta_{23} = \pi/4$. Whether they predict $+\pi/2$ or $-\pi/2$, additional information need to be provided[11]. It has been pointed out that the general form in eq.(6) is a necessary condition for $\delta = \pm\pi/2$ and $\theta_{23} = \pi/4$, but not sufficient condition. In our later discussions, unless specified, the mass matrix of the form in eq.(6) is always referred to a general form whose elements are not necessarily given by those in eq.(4). While the mass matrix in the form of eq.(7) provide sufficient condition for $\delta = \pm\pi/2$ and $\theta_{23} = \pi/4$ when s_{13} and $\sin \delta$ are not zero. The simplicity of the above mass matrix may serve as a good starting point to understand possible underlying theory. In fact it has been shown that the above neutrino mass matrix is a consequence of imposing a symmetry of the form $e \rightarrow e$, μ and τ exchange with CP conjugation discussed by Grimus and Lavoura in Ref.[8], which we will refer to as the Grimus-Lavoura symmetry (GLS).

In this work, we study realizations of $\delta = \pm\pi/2$ and $\theta_{23} = \pi/4$ in type-II seesaw model with A_4 flavor symmetry. Models based on A_4 symmetry has been shown to be able to provide a good scenario to achieve this[11, 12]. In A_4 models, the charged lepton mass matrix M_l is diagonalized from left (rotation on left-handed charged leptons) by the characteristic matrix U_l for A_4 symmetry model buildings[7],

$$M_l = U_l \hat{m}_l U_r, \quad U_l = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad (8)$$

where $\omega = \exp(i2\pi/3)$ and $\omega^2 = \exp(i4\pi/3)$. U_r is a unitary matrix, but does not play a role in determining V_{PMNS} . We will not specify its form here.

If neutrinos are Majorana particles, the most general mass matrix is of the form

$$M_\nu = \begin{pmatrix} w_1 & x & y \\ x & w_2 & z \\ y & z & w_3 \end{pmatrix}, \quad (9)$$

which can be diagonalized by unitary matrix V_ν , $M_\nu = V_\nu \hat{m}_\nu V_\nu^T$.

The mixing matrix V_{PMNS} is given by

$$V_{PMNS} = U_l^\dagger V_\nu . \quad (10)$$

In the basis where charged lepton is diagonalized, the neutrino mass matrix is of the form given by eq.(6) with[12]

$$\begin{aligned} a &= \frac{1}{3}(w_1 + w_2 + w_3 + 2(x + y + z)) , \\ d + i\gamma &= \frac{1}{3}(w_1 + \omega w_2 + \omega^2 w_3 + 2(\omega^2 x + \omega y + z)) , \\ d - i\gamma &= \frac{1}{3}(w_1 + \omega^2 w_2 + \omega w_3 + 2(\omega x + \omega^2 y + z)) , \\ c + i\beta &= \frac{1}{3}(w_1 + \omega^2 w_2 + \omega w_3 - \omega x - \omega^2 y - z) , \\ c - i\beta &= \frac{1}{3}(w_1 + \omega w_2 + \omega^2 w_3 - \omega^2 x - \omega y - z) , \\ b &= \frac{1}{3}(w_1 + w_2 + w_3 - (x + y + z)) . \end{aligned} \quad (11)$$

If one imposes the GLS on the neutrino mass matrix, all parameters in the set $P = (w_i, x, y, z)$ are dictated to be real, and will predict[11, 12] $\delta = \pm\pi/2$ and $\theta_{23} = \pi/4$. Therefore in A_4 model building for neutrino masses with $\delta = \pm\pi/2$ and $\theta_{23} = \pi/4$, it is essentially to make sure that U_l is of the form given by eq.(8) and require the resulting mass matrix to satisfy GLS. Note that in this case since the parameters in set P are all real, the complexity of the mixing matrix is purely due to the appearance of ω and ω^2 . When the GLS is broken by allowing the parameters in P can be complex, there are more sources for CP violation and the model does not predict $\delta = \pm\pi/2$ and $\theta_{23} = \pi/4$ automatically. This points a way to modify the predictions to fit data should future experiments will find δ and θ_{23} to be deviate significantly from $-\pi/2$ and $\pi/4$. We will study both cases with the parameters in set P to be real and complex in the rest of the paper.

II. TYPE-II SEESAW MODEL WITH A_4 SYMMETRY

We now construct a Type-II seesaw model[13] with A_4 family symmetry to realize the forms of mass matrices in eqs.(6) and (7). A different model based on Type-II seesaw with A_4 has been constructed to realized tribi-maximal neutrino mixing[14]. In our model, the left-handed lepton doublet l_L and right-handed charged lepton singlet l_R have the following standard $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge and A_4 family symmetry properties

$$l_L : (1, 2, -1)(3) , \quad l_R : (1, 1, -2)(1 + 1'' + 1') , \quad (12)$$

where the first three numbers in the first bracket indicate the $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ transformation properties. The numbers in the second bracket indicate the A_4 representations.

To obtain desired mixing pattern, the Higgs sector is enlarged to have two types of Higgs doublets, ϕ and Φ , and two triplets, Δ and χ for neutrino masses. They transform as

$$\phi : (1, 2, -1)(1) , \quad \Phi : (1, 2, -1)(3) , \quad \Delta^{0,1,2} : (1, 3, -2)(1 + 1' + 1'') , \quad \chi : (1, 3, -2)(3) . \quad (13)$$

The Lagrangian responsible for the lepton mass matrix is

$$L = y_e \bar{l}_L \tilde{\Phi} l_R^1 + y_\mu \bar{l}_L \tilde{\Phi} l_R^2 + y_\tau \bar{l}_L \tilde{\Phi} l_R^3 \\ + Y_\nu^0 \bar{l}_L \Delta^0 l_L^c + Y_\nu' \bar{l}_L \Delta' l_L^c + Y_\nu'' \bar{l}_L \Delta'' l_L^c + y_\nu \bar{l}_L \chi l_L^c + H.C. \quad (14)$$

If the structure of the vacuum expectation value (vev) is of the form $\langle \Phi_i \rangle = v_i^\Phi$, $\langle \chi_i \rangle = v_i^\chi$, $\langle \phi \rangle = v_\phi$, and $\langle \Delta^{0,'''} \rangle = v_\Delta^{0,'''}$, one obtains the charged lepton and neutrino mass matrices M_l and M_ν as

$$M_l = \sqrt{3} \begin{pmatrix} v_1^\Phi & 0 & 0 \\ 0 & v_2^\Phi & 0 \\ 0 & 0 & v_3^\Phi \end{pmatrix} U_l \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix}, \quad M_\nu = \begin{pmatrix} w_1 & x & y \\ x & w_2 & z \\ y & z & w_3 \end{pmatrix}, \quad (15)$$

where

$$w_1 = Y_\nu^0 v_\Delta^0 + Y_\nu' v_\Delta' + Y_\nu'' v_\Delta'', \\ w_2 = Y_\nu^0 v_\Delta^0 + \omega^2 Y_\nu' v_\Delta' + \omega Y_\nu'' v_\Delta'', \\ w_3 = Y_\nu^0 v_\Delta^0 + \omega Y_\nu' v_\Delta' + \omega^2 Y_\nu'' v_\Delta'', \\ x = y_\nu v_3^\chi, \quad y = y_\nu v_2^\chi, \quad z = y_\nu v_1^\chi. \quad (16)$$

If the vevs of $\langle \Phi_{1,2,3} \rangle = v_{1,2,3}^\Phi$ are all equal to v^Φ , the vev structure of the Higgs fields breaks A_4 , but left with a Z_3 residual symmetry generated by $\{I, c, a\}$. Here c, a are A_4 group elements defined in Ref.[15]. This will lead to the charged lepton mass matrix of the form

$$M_l = U_l \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \quad (17)$$

where $m_{e,\mu,\tau} = \sqrt{3} y_{e,\mu,\tau} v^\Phi$.

As long as Z_3 residual symmetry in the model is not broken, we have one of the key ingredients in realizing the form of charged lepton mass matrix in eq.(8). In the basis where the charged lepton mass matrix is diagonalized, the neutrino mass matrix will be the same form in eq.(6).

In the rest of this paper, we will study consequences related to the above mass matrix. We will study conditions on the model parameters imposed by GLS, and phenomenologically acceptable models can result in this class of models.

Before carrying out detailed analysis for neutrino mixing, we would like to point out that this model can easily accommodate data in the quark sector if one assigns the left- and right- handed quarks Q_L, U_R and D_R as A_4 singlet “1”. In this case the Yukawa couplings for quark masses and their mixing are given by

$$L = \bar{Q}_L Y_u \phi U_R + \bar{Y}_d \bar{Q}_L \tilde{\phi} D_R + H.C.. \quad (18)$$

This will give, in general, arbitrary 3×3 up- and down- quark mass matrices $M_{u,d}$ after ϕ develops a non-zero vev v_ϕ with $M_{u,d} = Y_{u,d} v_\phi$. These matrices will have no predictive power for quark masses and their mixing, but have no problem in accommodating experimental data.

III. MASS MATRIX PREDICTING $\delta = \pm\pi/2$ AND $\theta_{23} = \pi/4$

If the parameters in the set P are all real, the resulting mass matrix is of the form given in eq.(7). Therefore this model predicts

$$\delta = \pm\frac{\pi}{2}, \quad \theta_{23} = \frac{\pi}{4}. \quad (19)$$

The above prediction can also be obtained by studying the mixing matrix V_{PMNS} is eq.(10). For an arbitrary complex symmetric matrix M_ν , the matrix V_ν diagonalizes it, has the most general form

$$V_\nu = \text{diag}(1, e^{i\tau_2}, e^{i\tau_3})U(\tilde{\theta}_{12}, \tilde{\theta}_{13}, \tilde{\theta}_{23}, \tilde{\delta}). \quad (20)$$

Here we have absorbed a Majorana phase matrix $\text{diag}(1, e^{i\eta_2}, e^{i\eta_3})$ on the right of the above equation into the eigen-masses $m_i e^{i2\eta_i}$.

If the parameter set in P are all real, the mass matrix M_ν can be diagonalized by an orthogonal mixing matrix $V_\nu = U(\tilde{\theta}_{12}, \tilde{\theta}_{13}, \tilde{\theta}_{23}, \tilde{\delta} = 0)$, and $e^{i\tau_i}$ are real. Depending on conventions, here \tilde{s}_{ij} and \tilde{c}_{ij} need not to be normalized all to be positive. We have the V_{PMNS} given as the following

$$\begin{pmatrix} \frac{-\tilde{s}_{12}(\tilde{c}_{23}-\tilde{s}_{23})+\tilde{c}_{12}(\tilde{c}_{13}-\tilde{s}_{13}(\tilde{c}_{23}+\tilde{s}_{23}))}{\sqrt{3}} & \frac{\tilde{c}_{12}(\tilde{c}_{23}-\tilde{s}_{23})+\tilde{s}_{12}(\tilde{c}_{13}-\tilde{s}_{13}(\tilde{c}_{23}+\tilde{s}_{23}))}{\sqrt{3}} & \frac{\tilde{s}_{13}+\tilde{c}_{13}(\tilde{c}_{23}+\tilde{s}_{23})}{\sqrt{3}} \\ \frac{-\tilde{s}_{12}(\omega^2\tilde{c}_{23}-\omega\tilde{s}_{23})+\tilde{c}_{12}(\tilde{c}_{13}-\tilde{s}_{13}(\omega\tilde{c}_{23}+\omega^2\tilde{s}_{23}))}{\sqrt{3}} & \frac{\tilde{c}_{12}(\omega^2\tilde{c}_{23}-\omega\tilde{s}_{23})+\tilde{s}_{12}(\tilde{c}_{13}-\tilde{s}_{13}(\omega\tilde{c}_{23}+\omega^2\tilde{s}_{23}))}{\sqrt{3}} & \frac{\tilde{s}_{13}+\tilde{c}_{13}(\omega\tilde{c}_{23}+\omega^2\tilde{s}_{23})}{\sqrt{3}} \\ \frac{-\tilde{s}_{12}(\omega\tilde{c}_{23}-\omega^2\tilde{s}_{23})+\tilde{c}_{12}(\tilde{c}_{13}-\tilde{s}_{13}(\omega^2\tilde{c}_{23}+\omega\tilde{s}_{23}))}{\sqrt{3}} & \frac{\tilde{c}_{12}(\omega\tilde{c}_{23}-\omega^2\tilde{s}_{23})+\tilde{s}_{12}(\tilde{c}_{13}-\tilde{s}_{13}(\omega^2\tilde{c}_{23}+\omega\tilde{s}_{23}))}{\sqrt{3}} & \frac{\tilde{s}_{13}+\tilde{c}_{13}(\omega^2\tilde{c}_{23}+\omega\tilde{s}_{23})}{\sqrt{3}} \end{pmatrix} \quad (21)$$

The above matrix has the property that, $|V_{PMNS,2i}| = |V_{PMNS,3i}|$. Rewriting the above into the standard form of the mixing matrix, this leads to [9]

$$\theta_{23} = \frac{\pi}{4}, \quad s_{13} \cos \delta = 0. \quad (22)$$

Since experimentally $s_{13} \neq 0$, then $\cos \delta = 0$ which leads to $\delta = \pm\pi/2$.

The other angles can be determined by

$$s_{13} = |V_{13}|, \quad t_{12} = \frac{|V_{12}|}{|V_{11}|}, \quad (23)$$

where $t_{ij} = s_{ij}/c_{ij}$.

The condition whether δ takes $+\pi/2$ or $-\pi/2$ is determined by using the Jarlskog parameter [16] $J = c_{13}^2 s_{12} c_{12} s_{23} c_{23} s_{13} \sin \delta = \text{Im}(V_{11} V_{12}^* V_{21}^* V_{22})$ to obtain

$$\sin \delta = \frac{(1 - |V_{13}|^2) \text{Im}(V_{11} V_{12}^* V_{21}^* V_{22})}{|V_{11}| |V_{12}| |V_{23}| |V_{33}| |V_{13}|}. \quad (24)$$

Note that if $\tilde{\delta} \neq 0$ and/or $e^{i\tau_i}$ are not real, in general $|V_{PMNS,2i}|$ is not the same as $|V_{PMNS,3i}|$. θ_{23} and δ will not necessarily be $\pi/4$ and $\pm\pi/2$. One should be cautious about this. This uncertainty actually can be turned into a good use to provide a possible way to modify the predictions for θ_{23} and δ . The value for θ_{23} is determined by

$$t_{23} = \frac{|V_{23}|}{|V_{33}|}, \quad (25)$$

The angles θ_{13} and θ_{12} are determined by eq.(23), and the CP violating phase δ is given by eq.(24).

The above discussions provides a way of looking at how the reality of the parameters in the set P leads to $\delta = \pm\pi/2$ and $\theta_{23} = \pi/4$. It is essential to have the parameters in the set P to be all real. The complexity of the parameters can appear in the Yukawa couplings, in the vevs, and also in places where ω^i appear in neutrino mass matrix. One needs to see with what conditions they can be made real.

To make the Yukawa couplings and scalar vevs real, one can require the model Lagrangian to satisfy a generalized CP symmetry under which

$$(l_{e,L}, l_{\mu,L}, l_{\tau,L}) \rightarrow (l_{e,L}^{CP}, l_{\tau,L}^{CP}, l_{\mu,L}^{CP}), \quad \Phi = (\Phi_1, \Phi_2, \Phi_3) \rightarrow (\Phi_1^\dagger, \Phi_3^\dagger, \Phi_2^\dagger), \quad (26)$$

$$(\Delta^0, \Delta', \Delta'') \rightarrow (\Delta^{0\dagger}, \Delta'^\dagger, \Delta''^\dagger), \quad (\chi_1, \chi_2, \chi_3) \rightarrow (\chi_1^\dagger, \chi_3^\dagger, \chi_2^\dagger),$$

and all other fields transform the same as those under the usual CP symmetry. Here the superscript CP in the above indicate that the fields are the usual CP transformed fields.

The above transformation properties will transform relevant terms into their complex conjugate ones. Requiring the Lagrangian to be invariant under the above transformation dictates the Yukawa couplings to be real. The same requirement will dictate the scalar potential to forbid spontaneous CP violation and vevs to be real. One, however, notices that the parameters $w_{2,3}$ in eq.(16) are in general complex even if the Yukawa couplings and the vevs of the scalar fields are made real because the appearance of ω^i . To make them real to satisfy GLS, it is therefore required that

$$Im(\omega^2 Y'_\nu v'_\Delta + \omega Y''_\nu v''_\Delta) = Im(\omega Y'_\nu v'_\Delta + \omega^2 Y''_\nu v''_\Delta) = 0. \quad (27)$$

The above can be achieved by the absent of the scalar fields Δ'' in the theory or $Y'_\nu v_{\Delta'} = Y''_\nu v_{\Delta''}$. We will consider examples for each case in the next section.

From the above discussions, we know that in general the mass matrices obtained can accommodate values different than $\pi/4$ and $\pm\pi/2$ for θ_{23} and δ respectively, we will not carry out a full numerical search analysis for parameter spaces, but to take some simple cases to show how the modifications arise when GLS is broken and how future experiments can test this class of models. These models, due to additional constraints, will have some additional predictions than the general one.

A. Model A: Z_2 residual symmetry for neutrino mass matrix

If vev of χ_2 component of χ is non-zero, but the vevs of $\chi_{1,3}$ are zero, the vev structure breaks A_4 down to a Z_2 generated by $\{1, r_2\}$. Here r_2 is an A_4 group element defined in Ref.[15]. The charge lepton and neutrino mass matrices are given by

$$M_l = U_l \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \quad M_\nu = \begin{pmatrix} w_1 & 0 & y \\ 0 & w_2 & 0 \\ y & 0 & w_3 \end{pmatrix}. \quad (28)$$

The parameters in the set w_i , and y are in general complex.

If the fields, Δ' and Δ'' are absent from the model, one would have $w_1 = w_2 = w_3 = Y_\nu^0 v_\Delta^0$. The resulting mass matrices lead to the well known tribi-maximal mixing[17] which had been the focus for A_4 symmetry model buildings before[15, 18–20]. In this case, $\theta_{13} = 0$ and phase δ is zero. Since

experimentally s_{13} has been determined to be non-zero, this model is ruled out. Modifications have to be implemented[21–23] to accommodate data. A way out is to keep $Y'_\nu v_{\Delta'} = Y''_\nu v_{\Delta''}$ in the model as discussed in the previous section.

We now summarize the main results follow Ref.[11] for this model. This also services as an outline how analysis can be carried out for relevant models. Letting M_ν be diagonalized by \tilde{V}_ν , one has

$$M_\nu = \tilde{V}_\nu \hat{m}_\nu \tilde{V}_\nu^T . \quad (29)$$

\tilde{V}_ν in this case can always be written in the following way with Majorana phases to be absorbed in to the neutrino masses m_i ,

$$\tilde{V}_\nu = V_\rho V_\nu , \quad \text{with} \quad V_\nu = \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix} , \quad (30)$$

where V_ρ is a diagonal matrix $\text{diag}(1, 1, e^{i\rho})$ with $\tan \rho = \text{Im}(yw_1^* + y^*w_3)/\text{Re}(yw_1^* + y^*w_3)$. $s = \sin \theta$ and $c = \cos \theta$.

Expressing the mixing angle θ in terms of the model parameters, we obtain

$$\tan 2\theta = \frac{2|yw_1^* + w_3y^*|}{|w_1|^2 - |w_3|^2} . \quad (31)$$

Using $\hat{m}_\nu = V_\nu^\dagger V_\rho^\dagger M_\nu V_\rho^* V_\nu^*$ derived from eq.(29), we obtain the Majorana phases α_i of m_i in terms of the model parameters as

$$\alpha_{1,3} = \text{Arg}(w_i(1 \pm \cos 2\theta) + w_2 e^{-i2\rho}(1 \mp \cos 2\theta) \pm 2 \sin 2\theta y e^{-i\rho}) , \quad \alpha_2 = \text{Arg}(w_2) . \quad (32)$$

One can always normalize the Majorana phase α_2 to be zero without of generality. In this basis, the phase of w_2 is also zero. Although the Majorana phases can be expressed in terms of the model parameters, since there are more number of parameters than the mixing angles and eigen-masses, there is no prediction for the Majorana phases. We will not discuss them in our later numerical analysis any more.

The mixing matrix V_{PMNS} can be, in general, written as

$$V_{PMNS} = U_l^\dagger V_\rho V_\nu = \frac{1}{\sqrt{3}} \begin{pmatrix} c + se^{i\rho} & 1 & ce^{i\rho} - s \\ c + \omega se^{i\rho} & \omega^2 & \omega ce^{i\rho} - s \\ c + \omega^2 se^{i\rho} & \omega & \omega^2 ce^{i\rho} - s \end{pmatrix} , \quad (33)$$

Using eqs.(23), (24) and (25), we find that

$$s_{12} = \frac{1}{\sqrt{2}(1 + cs \cos \rho)^{1/2}} , \quad s_{23} = \frac{(1 + cs \cos \rho + \sqrt{3}cs \sin \rho)^{1/2}}{\sqrt{2}(1 + cs \cos \rho)^{1/2}} , \quad s_{13} = \frac{(1 - 2cs \cos \rho)^{1/2}}{\sqrt{3}} . \quad (34)$$

and

$$\sin \delta = (1 + \frac{4c^2 s^2 \sin^2 \rho}{(c^2 - s^2)^2})^{-1/2} (1 - \frac{3c^2 s^2 \sin^2 \rho}{(1 + cs \cos \rho)^2})^{-1/2} \times \begin{cases} -1 , & \text{if } c^2 > s^2 , \\ +1 , & \text{if } s^2 > c^2 . \end{cases} \quad (35)$$

From the above, one clearly sees that if $\sin \rho$ is not zero, $|\delta|$ and θ_{23} deviate from $\pi/2$ and $\pi/4$, respectively. In the limit ρ goes to zero, however, the above recovers the results with real parameter set P with $\delta = \pm\pi/2$ and $\theta_{23} = \pi/4$.

There are two interesting features for this model worth mentioning. One of is that $|V_{e2}|$ to be $1/\sqrt{3}$ which agree with data. s_{12} is always larger or equal to $1/\sqrt{3}$ which is a decisive test for this model. Another is that although the Dirac phase δ depends on the phase ρ , the Jarlskog parameter J which is independent of ρ given by $J = -(c^2 - s^2)/6\sqrt{3}$. This implies that CP violation related to neutrino oscillation is still purely due to intrinsic CP violation. This model can be made in agreement with data at 1σ level.

If $\rho = 0$ and $c = s = 1/\sqrt{2}$, the mixing pattern is the tribi-maximal. However, if ρ is not zero, even if $c = s = 1/\sqrt{2}$, s_{13} can be non-zero, s_{12} and s_{23} are also modified from their tribi-maximal values

$$s_{12} = \frac{1}{(2 + \cos \rho)^{1/2}}, \quad s_{23} = \frac{1}{\sqrt{2}} \left(1 + \frac{\sqrt{3} \sin \rho}{2 + \cos \rho}\right)^{1/2}, \quad s_{13} = \frac{(1 - \cos \rho)^{1/2}}{\sqrt{3}}. \quad (36)$$

J is exactly zero which implies $\sin \delta = 0$.

B. Model B: mass matrix with $w_{1,2,3} = 0$

Without $\Delta^{0',''}$, $w_{1,2,3} = 0$. In this case all the three vevs of χ should not be equal in order to fit data on the neutrino mass-squared differences $\Delta m_{21}^2 = |m_2|^2 - |m_1|^2$ and $\Delta m_{31}^2 = |m_3|^2 - |m_1|^2$. The Z_2 residual symmetry mentioned previously is also broken in this case. If CP is spontaneously broken in the Higgs potential, v_i^X may be complex too. Therefore the parameters in the set P : $\{x, y, z\} = \{|x|e^{i\sigma_x}, |y|e^{i\sigma_y}, |z|e^{i\sigma_z}\}$ are in general complex. One can always choose, without loss of generality, to rewrite the neutrino mass matrix as $M_\nu = V_\sigma \tilde{M}_\nu V_\sigma^T$

$$\tilde{M}_\nu = \begin{pmatrix} 0 & |x| & |y| \\ |x| & 0 & |z| \\ |y| & |z| & 0 \end{pmatrix}, \quad V_\sigma = e^{i\sigma_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\sigma_2} & 0 \\ 0 & 0 & e^{i\sigma_3} \end{pmatrix}, \quad (37)$$

where $\sigma_1 = (\sigma_z - \sigma_x - \sigma_y)/2$, $\sigma_2 = \sigma_z - \sigma_y$ and $\sigma_3 = \sigma_x - \sigma_y$. σ_1 can be absorbed into lepton fields. We will neglect it in the following analysis.

Let us discuss now how the model parameters can be determined by data. First of all since \tilde{M}_ν is a real matrix, the eigen-values \tilde{m}_i are all real and $\tilde{m}_1 + \tilde{m}_2 + \tilde{m}_3 = 0$. The absolute value of neutrino mass $|m_i| = |\tilde{m}_i|$ can be solved as functions of Δm_{21}^2 and Δm_{31}^2 . For the values allowed by experimental data, the solutions are given by up to a overall sign

$$\begin{aligned} \tilde{m}_1 &= \frac{(\Delta m_{21}^2 - \Delta X^2) \sqrt{\Delta m_{21}^2 - \Delta m_{31}^2 + 2\Delta X^2}}{\sqrt{3}(\Delta m_{21}^2 + \Delta m_{31}^2)} \\ \tilde{m}_3 &= \frac{(\Delta m_{31}^2 + \Delta X^2) \sqrt{\Delta m_{21}^2 - \Delta m_{31}^2 + 2\Delta X^2}}{\sqrt{3}(\Delta m_{21}^2 + \Delta m_{31}^2)} \\ \Delta X^2 &= \sqrt{(\Delta m_{21}^2)^2 + \Delta m_{21}^2 \Delta m_{31}^2 + (\Delta m_{31}^2)^2} \end{aligned} \quad (38)$$

One can easily check that multiplying a “-” to the above one obtains another solution. The overall sign can be absorbed into lepton field phases redefinition. We will use the above normalization for

the overall sign. In the numerical study later, we will show that only inverted hierarchy is allowed. Therefore, $m_{1,3} > 0$ and $m_2 < 0$. Since the eigen-masses are all real, there is no Majorana phases in this model.

Since \tilde{M}_ν is real, it can be diagonalized by a orthogonal matrix. Assuming $U(\tilde{\theta}_{13}, \tilde{\theta}_{12}, \tilde{\theta}_{23}, \tilde{\delta} = 0)$ diagonalizes \tilde{M}_ν , one can express the fact that the “11” and “22” entries are zero as

$$\begin{aligned} \tilde{c}_{13}^2(\tilde{m}_1\tilde{c}_{12}^2 + \tilde{m}_2\tilde{s}_{12}^2) + \tilde{m}_3\tilde{s}_{13}^2 &= 0, \\ \tilde{m}_1(\tilde{s}_{13} - \tilde{t}_{12}\tilde{t}_{23})(\tilde{t}_{12} + \tilde{t}_{23}\tilde{s}_{13}) - \tilde{m}_2(\tilde{t}_{12}\tilde{s}_{13} + \tilde{t}_{23})(1 - \tilde{t}_{12}\tilde{t}_{23}\tilde{s}_{13}) + \tilde{m}_3\frac{\tilde{c}_{13}^2}{\tilde{c}_{12}}\tilde{t}_{23} &= 0, \end{aligned} \quad (39)$$

where $\tilde{t}_{ij} = \tilde{s}_{ij}/\tilde{c}_{ij}$.

From the above two equations, one can express \tilde{s}_{12} and \tilde{s}_{23} as functions of \tilde{s}_{13} and \tilde{m}_i . Since \tilde{m}_i are known, the elements in $V_{PMNS} = U_l^\dagger V_\sigma V_\nu$ can be expressed as functions of σ_2 , σ_3 and \tilde{s}_{13} . Using the relations between the elements V_{ij} of V_{PMNS} with s_{ij} and δ in eqs.(23) and (24), one can vary σ_2 , σ_3 and \tilde{s}_{13} to see if the resulting values for $s_{12,13,23}$ are in the allowed values, and obtain δ by eq.(24).

Similar analysis had been used to rule out[24] the simple version of Zee model where the mass matrix for neutrinos is \tilde{M}_ν in the diagonalized basis of the charged leptons. Here the appearance of U_l may save the model. We find that, unfortunately, that with real parameters in P which implies $\sigma_{2,3}$ to be zero, δ and θ_{23} are predicted to be $-\pi/2$ and $\pi/4$, but there is no solutions within the allowed ranges which can predict s_{12} to be consistent with data. With complex parameters in P , the values for $s_{12,13,23}$ can be in agreement with data, it is, however, not possible to get δ to be close to $-\pi/2$. More numerical details will be provided in the next section.

IV. NUMERAL ANALYSIS

In this section, we compare experimental data with the model predictions for the mixing angles and CP violating phase. There are several global fits of neutrino data[3, 4]. The latest fit gives the central values, 1σ errors and the 2σ ranges as the following[3]

	δ/π	s_{12}^2	$s_{13}^2/10^{-2}$	s_{23}^2	
<i>NH</i>	$1.41_{-0.44}^{+0.55}$	0.323 ± 0.016	2.26 ± 0.12	$0.567_{-0.124}^{+0.032}$	
2σ region	$0.0 \sim 2.0$	$0.292 \sim 0.357$	$2.02 \sim 2.50$	$0.414 \sim 0.623$	(40)
<i>IH</i>	1.48 ± 0.31	0.323 ± 0.016	2.29 ± 0.12	$0.573_{-0.039}^{+0.025}$	
2σ region	$0.00 \sim 0.09 \& 0.86 \sim 2.0$	$0.292 \sim 0.357$	$2.05 \sim 2.52$	$0.435 \sim 0.621$	

and the corresponding values for mass-squared differences are given by

	$\Delta m_{21}^2 [10^{-5}] eV^2$	$ \Delta m_{31}^2 [10^{-3}] eV^2$	
<i>NH</i>	$7.60_{-0.18}^{+0.19}$	$2.48_{-0.07}^{+0.05}$	
2σ region	$7.26 \sim 7.99$	$2.35 \sim 2.59$	(41)
<i>IH</i>	$7.60_{-0.18}^{+0.19}$	$2.38_{-0.06}^{+0.05}$	
2σ region	$7.26 \sim 7.99$	$2.26 \sim 2.48$	

Here *NH* and *IH* indicate neutrino mass hierarchy patterns of normal hierarchy and inverted hierarchy, respectively. We will use the above data for comparison.

A. Some general comments

The model discussed achieved a natural way to have the CP violating angle δ to be $\pm\pi/2$ and θ_{23} to be $\pi/4$. The condition is to have all the parameters in the set P to be real. This requirement can be taken as a consequence of having CP violation only caused by intrinsic source of CP violation. The condition for choosing $\delta = -\pi/2$ can be determined using eq.(24) which requires $Im(V_{11}V_{12}^*V_{21}^*V_{22})$ to be negative.

The value $-\pi/2$ predicted in the model is in agreement with IH within 1σ range. Although for NH case δ is outside of 1σ range, there is no problem with 2σ range. For s_{23} , the model predicts $s_{23}^2 = 0.5$. This value is outside of 1σ range for both the NH and IH cases. However, they are, again, in agreement with data within 2σ . Other parameters, Δm_{ij}^2 , s_{13} and s_{12} can be easily fitted to within 1σ level of current experimental data.

With complex parameters in P , $\delta = -\pi/2$ and $\theta_{23} = \pi/4$ can also deviate away from $-\pi/2$ and $\pi/4$ with $\sin\delta$ determined by eq.(24) and θ_{23} determined by eq.(25). One can find solutions where $\theta_{12,13,23}$ and δ to take the central values from current data.

As the general model is expected to be able to fit data, it may be more instructive to analyze some simplified versions than just providing with numbers. We provide more details of Model A and Model B discussed earlier next to see how additional assumptions restrict the level of model agreement with data.

B. Model A predictions

With real values for w_i and y in Model A, by adjusting the values, w_i , and y both NH and IH mass patterns can be obtained. The predictions for δ and θ_{23} are $\pm\pi/2$ and $\theta_{23} = \pi/4$, just like the general case discussed in the previous section. Since δ should be close to $-\pi/2$, one should take the parameter space so that $c^2 > s^2$. In the model $s_{13} = (1 - 2cs)^{1/2}/\sqrt{3}$ is not predicted. But one can use information from s_{13} to fix $cs = 0.497 \pm 0.018$ to predict $s_{12}^2 = 0.334 \pm 0.004$ for both NH and IH cases. This is in agreement with data within 1σ . Note that $V_{e2}^2 = (s_{12}c_{13})^2 = 1/3$. It agrees with data within 1σ .

It is remarkable that neutrino mixing matrix in this model with just one free parameter can be in reasonable agreement with data. This may be a hint that it is the form for mixing matrix, at least as the lowest order approximation, that a underlying theory is producing. One should take this mass matrix seriously in theoretical model buildings.

If the parameters in the set P are complex, and therefore a new phase ρ appears in the model. In this case, the new parameter ρ can be used to improve agreement of the model with data. In both NH and IH cases, δ and s_{23} can be brought into agreement with data at 1σ level. As an example, we take the largest value of cs so that s_{13} takes its lower 1σ allowed value, and then varying $\cos\rho$ to obtain the upper 1σ allowed value. This fixes cs and $\cos\rho$ to be 0.468 and 0.992, respectively. With these values, s_{23} and δ are determined to: 0.534 and 1.426π , respectively. These values are in agreement with data at 1σ level.

For the case with $c = s$, the model is more restrictive. In this case $\sin\delta = 0$. With more precise data on the CP violating angle δ , this may rule out this simple case with high confidence level. If $\rho = 0$, the model is already ruled out at high precision from s_{13} measurement. However, with a non-zero ρ , the mixing angles can still be made to in agreement at 2σ level. In Fig. 1 we show s_{12} ,

s_{23} , s_{13} as functions of ρ . When chose $\cos\rho = 0.93$, we can get $s_{12} = 0.58, s_{23} = 0.78, s_{13} = 0.15$ which agree with the experimental data within 2σ range.

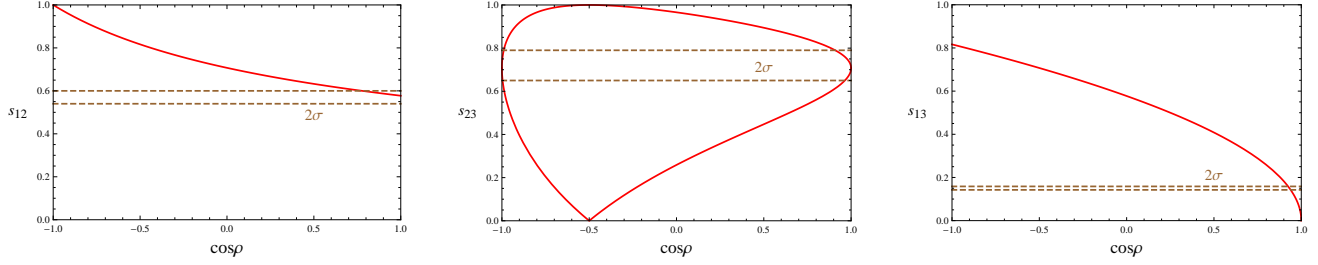


FIG. 1: s_{12} , s_{23} , s_{13} as functions of $\cos\rho$ with $c = s = \sqrt{2}/2$ compared with allowed ranges for (s_{13}, s_{23}, s_{12}) at 1σ and 2σ , respectively, given by $(0.147 \sim 0.155, 0.731 \sim 0.773, 0.554 \sim 0.582)$ and $(0.143 \sim 0.159, 0.660 \sim 0.788, 0.540 \sim 0.597)$.

More precise experimental data are required to distinguish the model with complex model parameters from that with the real parameters and other models, or to rule out the above simples completely.

C. Model B predictions

With real parameters in P in this model, the predictions, δ and θ_{23} are the same as the general case. Using eq.(39), one can choose \tilde{s}_{13} to be the parameter to fit all data. We find that no solutions can simultaneously bring s_{13} and s_{12} to be in 2σ allowed region compared with data. In Fig. 2 we show s_{13} , s_{12} as functions of \tilde{s}_{13} . Here, when s_{13} agree with the data in 2σ range, the allowed region for \tilde{s}_{13} are $(-0.594 \sim -0.565) \& (-0.012 \sim -0.010) \& (0.010 \sim 0.011)$, but when s_{12} agrees with the data in the 2σ ranges, the allowed region for \tilde{s}_{13} are only $(0.806 \sim 0.774)$, they have no overlap region for \tilde{s}_{13} . This model is therefore ruled out.

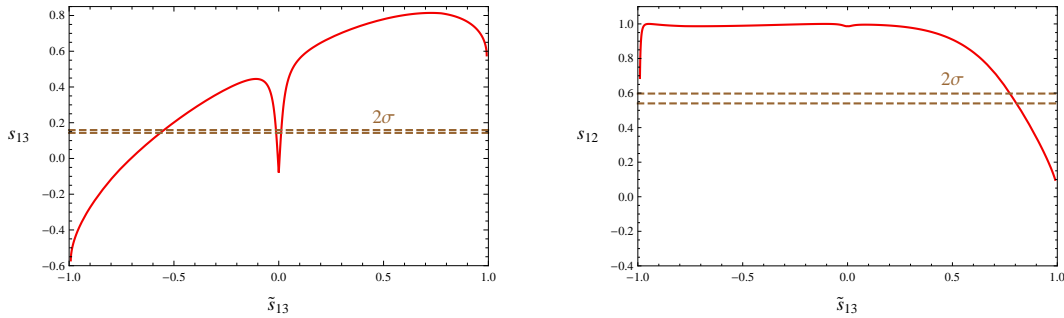


FIG. 2: s_{13} , s_{12} as functions of \tilde{s}_{13} .

With complex parameters in P , there are solutions of \tilde{s}_{13} and σ_i so that to make s_{23} , s_{13} and s_{12} to be consistent with data at 1σ level. However this requires one of $|\sin\sigma_i|$ to be away from zero which alter the prediction for δ significantly away from $-\pi/2$, although θ_{23} is still close to $\pi/4$. In Fig. 3, we scan the parameter space $-1 \leq \tilde{s}_{13} \leq 1, 0 \leq \sigma_2 \leq 2\pi, 0 \leq \sigma_3 \leq 2\pi$, and obtained the corresponding values of s_{12} , s_{23} , s_{13} that agree with data within 1σ and 2σ region respectively. And

in 1σ range, the CP phase will be constrained in $\sin\delta \sim (-0.5, 0.5)$, but in 2σ region, CP phase is free and can be in the range of 0 to 2π .

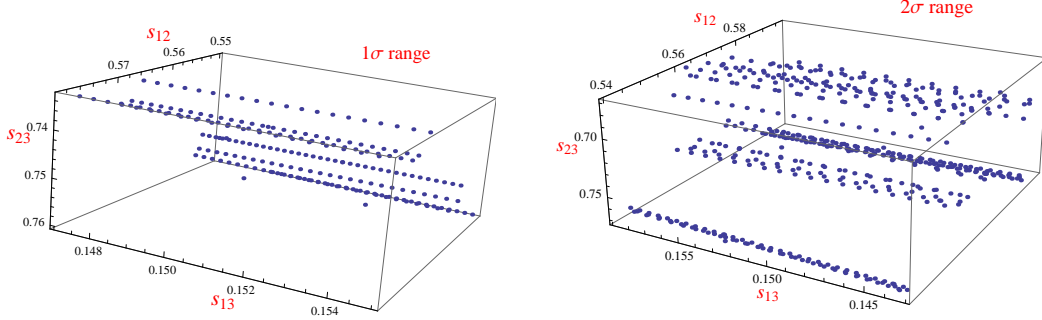


FIG. 3: The scanning points of the three free parameters in 1σ and 2σ region respectively.

For example with $\tilde{s}_{13} = 0.57$, $\sigma_2 = 7\pi/5$, $\sigma_3 = 26\pi/25$, one can obtain that $s_{13} = 0.15$, $s_{23} = 0.74$, $s_{12} = 0.57$. These mixing angles are within the 1σ allowed region, but it will have a CP phase $\delta = 1.06\pi$ significantly away from $3\pi/2$. A precise determination of the CP violating phase is needed to rule out the above model.

V. CONCLUSIONS

We have constructed theoretical models for neutrino mass matrix in Type-II seesaw with A_4 family symmetry. The models we constructed naturally predict that the CP violating phase δ is equal to $\pm\pi/2$ and at the same time the mixing angle θ_{23} is $\pi/4$. The reality of the parameters can be achieved by imposing a generalized CP symmetry in the Grimus-Lavoura symmetry limit. When the generalized CP symmetry is explicitly broken, the Yukawa couplings can be complex, the model predictions for δ and θ_{23} can be significantly modified. Two simple scenarios, Model A and Model B, are analyzed in detail to show how the modifications arise and how future experimental data can test this class of models.

The Model A has a characteristic prediction that $|V_{e2}| = 1/\sqrt{3}$. This model can accommodate experimental data at 1σ level. It can be taken as the lowest order neutrino mass matrix for future theoretical model buildings. When Yukawa couplings are complex, the CP violating phase δ and the mixing angle θ_{23} can be away from $-\pi/2$ and $\pi/4$. The crucial test for this model is to measure whether $|V_{e2}| = 1/\sqrt{3}$ holds to high precision.

In Model B, the diagonal entries of neutrino mass matrix are all zero in the weak interaction basis. This implies that the neutrino masses can be determined by the known neutrino mass-squared differences. We find that this model can only accommodate inverted neutrino mass hierarchy. The mixing angles s_{12} , s_{13} and s_{23} cannot simultaneously be in agreement with data at 2σ level if the Yukawa couplings are all real. With complex Yukawa couplings, the mixing angles can be brought into agreement with data at 1σ level, but the CP violating angle δ will be significantly away from $-\pi/2$. This provides a crucial test for this model.

At present, experimental data on neutrino mixing seem to hint that δ and θ_{23} to be close to $-\pi/2$ and $\pi/4$. Theoretical models which can naturally achieve such predictions are interesting to

study. The models we have constructed have many novel properties and can be tested. We will have to wait future experimental data to tell us more whether this class of neutrino mass matrix will survive.

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